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THE STRUCTURE OF EVOLUTIONAL JUMPS IN REVERSIBLE SYSTEMS†

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A general theory for jump structures in reversible symmetric systems is developed. It is established that, as in systems with dissipation [1–3], the type of jump structure depends on the number of intersections of the dispersion curve and the straight line corresponding to the jump velocity, while jumps for which a structure exists turn out to be at the same time evolutional jumps. The theory is applicable in cases when the dispersive properties of the medium prevail over its dissipative properties.

The basic principles of the theory of jumps in systems with dissipation are as follows [1]. One has a certain complete system of equations taking the dissipative terms into account. This simplified system is non-linear and of hyperbolic type; it does not always have continuous solutions, and jumps must be introduced. For a jump to be stable, a certain condition, known as evolutionality, must be satisfied: the number of boundary conditions for the jump must exceed by one the number of characteristics issuing from the jump. For a jump to exist, the complete system must have steady-state continuous solutions that describe jump-like transitions between uniform states; these solutions are known as discontinuity structures. The initial system of n equations may be written in conservative form, and its integral form yields n boundary conditions for the jump (principal boundary conditions). It has been shown for a large class of systems [2, 3] that the existence of a jump structure implies its evolutionality, that is, all necessary additional boundary conditions, if such are needed, may be obtained by an analysis of the jump structure.

All these principles have been developed in jump theory for non-dissipative systems, but assuming the presence of dispersion, which is responsible for the formation of the jump structure. This theory has the following essential differences.

1. By jumps one understands jumps between any periodic, quasi-periodic, stochastic and not only uniform states, and accordingly structures of such jumps prove to be more complex [4].
2. The simplified system of equations may be understood as a system of averaged equations for periodic or quasi-periodic states, and accordingly the number of characteristics in the system may exceed the number of unknowns in the complete system [5].
3. The structures of some jumps cease to exist and new types of jump, which do not exist in purely dissipative systems [6], appear. For example, in the case of non-dissipative models, for the most common jumps, in which the number of boundary conditions is n (shock waves in gas dynamics), there is no jump structure. These jumps are usually replaced by jumps between a uniform state and a periodic state. On the other hand, in dispersive systems, kink-type jumps ($n + 1$ conditions) become possible.
4. Jumps with a soliton structure appear for which the conservation laws cannot be used to obtain the boundary conditions.
5. In some cases, the number of independent boundary conditions obtainable from the conservation laws exceeds the number of initial equations [6].

The range of applicability of the non-dissipative theory of jumps may be any physical models in which the dispersive properties prevail over the dissipative properties, for example, jumps on the surface of an ideal liquid or jumps in plasma [7]. Jumps in a plasma are traditionally considered to be non-dissipative. Special mention should be made of the theory of jumps in wave models, which may be characterized in mathematical terms as models of the type of non-linear geometrical optics [4–6]. In wave jump theory it makes sense to speak only of non-dissipative jumps, since as yet one knows of no physical effects that might attribute to such models a property describable as "wave dissipation".

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1. JUMPS IN NON-DISSIPATIVE SYSTEMS

Let us consider the dispersive analogue of a jump for the equation

$$a_t + aa_x + b_3a_{xxx} = 0$$

This equation may have non-steady-state solutions of an oscillatory type and wave zones (WZs) that tend to a_1 and a_2 as $x \to \pm\infty$. As $t \to +\infty$ at one of the WZ boundaries, the first wave is approximately a solitary wave, followed by waves with amplitudes that decrease smoothly to zero. The length of the WZ increases linearly with time. Using Whitham's averaging method [8] to describe the evolution of the WZ, one derives a hyperbolic system in three unknowns which has self-similar solutions depending on x/t [7]. We shall call these solutions the self-similar (expanding) structure of a non-dissipative jump (another term: the transient structure of a collisionless shock wave [7], arises from the terminology used in plasma physics). If one compares these solutions with the analogous solutions of the Burgers equation, it turns out that this self-similar structure replaces the steady-state jump structure. For the Burgers equation, the wave velocity is $U = (a_1 + a_2)/2$, and in the given case the velocity of the leading edge of the WZ is $(2a_2 + a_1)/3$. In some cases, e.g. for practical applications, this information is quite sufficient; this might be termed the macroscopic approach. In this paper, a "jump" will mean a jump in wave amplitude at the WZ boundary. We shall formally consider a solitary wave at the WZ boundary as the steady-state structure of a non-dissipative jump. The system may be described on one side of the jump by the system of averaged equations for the wave zone, and on the other, by the usual simplified equation $a_t + aa_x = 0$.

In non-dissipative models with a complicated non-linearity, jumps appear without a WZ, and these, as in the case of dissipative jumps, may be rigorously treated as discontinuous solutions of the simplified system. However, the number of outgoing characteristics in these jumps is different. In models with complicated dispersion, steady jumps of non-soliton type with a WZ (jumps with radiated waves) appear. In that case the approach is the same as in the case of dissipative models, but the role of the simplified equations is played for the WZ by the equations obtained by averaging the complete system. In that context, "wave structures" are understood as solutions of the complete system which are steady in a certain coordinate system and become uniform, periodic or quasi-periodic states as $x \to +\infty$ and $x \to -\infty$.

A common property of non-dissipative models is that they are described by a symmetric, reversible, conservative system of equations. Integrating once, one obtains a dynamical system of ordinary differential equations that describe steady-state solutions. Introducing additional unknowns, one can present this system as a system of equations $u_{qx} = F_q(\mathbf{u})$ which are invariant under the transformation $x \to -x$, $u_q \to u_q$ for $q = 1, \ldots, 2n - 1$ (symmetric unknowns), $u_q \to -u_q$ for $q = 2, \ldots, 2n$ (anti-symmetric unknowns). This procedure is feasible for all physically meaningful time-reversible systems and is used to prove the existence of solitary and generalized solitary waves by the central-manifold method [9].

2. JUMP STRUCTURES

All the boundary conditions which are taken into account by a single integration of the equations of the initial conservative system are referred to as the principal conditions. It has been shown that in models with dissipation [2, 3], all additional boundary conditions necessary for evolutionality may be obtained by analysing the jump structures. In non-dissipative models, the number of additional conditions may be large, since, apart from the unknowns of the initial system, the computation must also include additional unknowns defining the amplitudes and wavelengths of the radiated waves: each radiated wave generates two additional boundary conditions. On the other hand, if there are further conservation laws in the initial system, besides those already used in its integration, they provide an alternative way of obtaining boundary conditions. In that case, if it is known that a jump exists and there are enough of these laws, an analysis of the discontinuity structure may be avoided [6].

Two approaches are presented to analyse the possibility that discontinuities exist, one geometric and the other algorithmic. Both approaches yield identical results. The second is essentially an informal presentation of the ideas of the first, by pointing out how solutions may be found by varying the initial data.

The geometric approach is based on the fact that, if the sum of the dimensions of two subspaces equals the dimension of the whole space, such subspaces may, in the standard case, intersect in one, finitely many or denumerably many points. But if the two subspaces intersect in such a way that they must have a common tangent vector, and the sum of their dimensions exceeds that of the space by one, then the standard position is that the subspaces may intersect along a curve. That is the situation if the two subspaces are generated by certain sets of phase curves of a dynamical system. We shall call such subspaces *phase subspaces*, introducing special notation for them. Suppose the subspace in question is generated by the phase trajectories $\{\mathbf{u}(x), -\infty < x < +\infty\}$ that pass through the ε-neighbourhood of a point C at $x = x_0$ and remain bounded or periodic at $+\infty$. We shall refer to such subspaces by notation of the form $S(C, \varepsilon;\ |\mathbf{u}| < M, x \to +\infty)$ or $S(C, \varepsilon;\ \mathbf{u}(x + T) \to \mathbf{u}(x), x \to +\infty)$. The quantity x_0 plays no role, because a solution is defined only apart from the phase shifts.

In the algorithmic approach, one analyses the correspondence between the number of free parameters of the initial data and the number of constraints imposed when the problem of seeking steady-state solutions is formulated. If the number of free parameters at the initial point of the phase space (the state on one side of the jump) is identical with the number of constraints imposed at the terminal point (the state on the other side of the jump), then the problem may have a solution, and it may be regarded as well-posed in the sense that one, finitely many or denumerably many solutions may exist. For symmetric soliton-like structures, solutions are sought somewhat differently, as well be described below.

For the versions of these systems linearized about uniform states, let us consider the dispersion curve (DC) $\omega = \omega(k)$ (it is assumed that this relation is obtained by the substitution $\sim\exp[i(kx - \omega t)]$) and the equation $R(U, k) = 0$ for the wavelengths of the steady-state solutions (where U is the jump velocity). The equation $R(U, k) = 0$ has $2n$ roots, and they are such that $k_{2m-1} = -k_{2m}$, $m = 1, \ldots, n$. A solution of the linearized dynamical system depends on $2n$ parameters c_j and may be written as $\mathrm{Re}(\Sigma_{j=1}^{2n} c_j \exp ik_j x)$. With wave numbers with $\mathrm{Im}(k_j) > 0$ one associates waves that increase as $x \to -\infty$, with $\mathrm{Im}(k_j) < 0$ one associates waves that increase as $x \to +\infty$, and with real wave numbers k_j one associates purely periodic waves. We will assume in what follows that the given qualitative properties are preserved in the non-linear versions of the systems being considered, with the difference that increasing waves may turn out to increase only in the neighbourhood of an equilibrium point and to remain bounded as they travel away from it.

Suppose the jump is moving at some velocity U. Considering the (ω, k) plane, draw the straight line $U = \omega/k$ corresponding to the jump velocity. To analyse whether the problem is well posed, it is important to know how many times this straight line intersects the DC. Counting theses intersections, we shall not include the intersection at the origin, since it corresponds to the multiple $k = 0$, already taken into account as an integration constant of the initial system. In addition, given the symmetry of the DC, only intersections with $k > 0$ will be counted. Each intersection yields two real roots, with opposite signs, of the equation $R(U, k) = 0$. The DC is different on different sides of the jump, so that the number of intersections on either side may also differ. Figure 1 is a schematic representation of the different possible relative positions of the straight line and the DC on either side of the jump, for states 1 and 2 (the curve for state 2 is the solid curve), on the assumption that the phase velocity is close to the characteristic velocity of the simplified system of equations. Corresponding to the tangent to the DC at $k = 0$ is a certain characteristic velocity; characteristics with this property will be called principal characteristics. For simplicity, the situation shown is one in which the DC has only one branch, but the conclusions reached below also hold for cases in which there are several branches. Figure 2 is a schematic

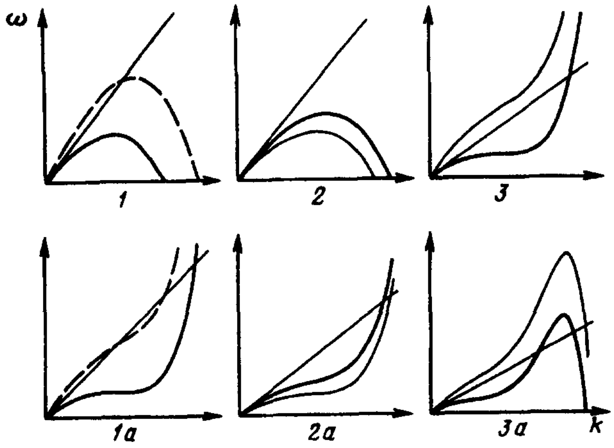

Fig. 1.

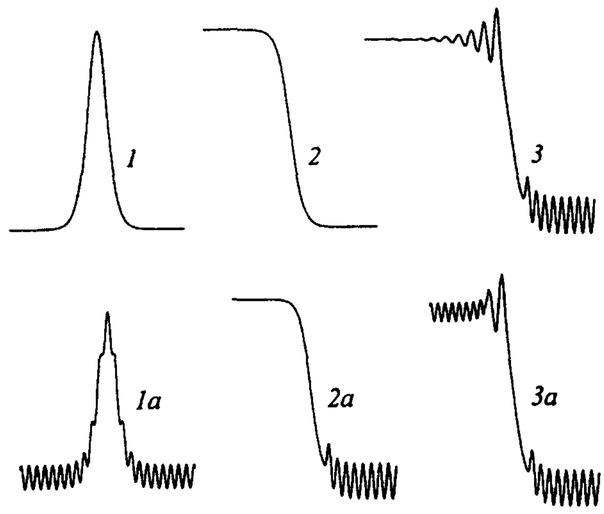

Fig. 2.

representation of the graphs of the corresponding steady-state solutions for unknowns of symmetric type.

Below we will analyse all possible jump structures, showing that if a jump structure exists, the jump will also be evolutional. Jump structures and their evolutionality will be analysed by induction, proceeding from the simplest structures, with the minimum number of radiated waves, to more complex structures, with subsequent addition of further waves. The main idea is as follows. Whether the principal characteristic being considered is incoming or outgoing depends on whether the straight line $U = \omega/k$ lies above that branch in the neighbourhood of the origin (the first factor). But this also determines the number of intersections of the straight line with the DC. The number of intersections determines the number of additional boundary conditions that arise when the structure is analysed (the second factor) and the number of additional outgoing characteristics associated with radiated waves (the third factor). Striking a balance among these three factors, one can ascertain whether the evolutionality condition is satisfied.

1. *A solitary wave*. Suppose the straight line does not intersect the DC. Then there are no real roots for the equilibrium point C_1 under consideration. There are n roots with $\mathrm{Im}(k) < 0$ (the waves increase as x increases) and n roots with $\mathrm{Im}(k) > 0$ (decreasing waves). Consider the subspaces $S_1 = S(C_1, \varepsilon; \mathbf{u} \to C_1, x \to -\infty)$ and $S_2 = \{\mathbf{u}, u_{2i} = 0\}$, both of dimension n. Since the sum of their dimensions is $2n$, there may, in general, be one intersection, finitely many or denumerably many. By the symmetry of the equations, such an intersection defines a solitary wave, and the point of intersection corresponds to the crest of the wave.

An algorithmic approach. If we have to find a solution with a constant at $-\infty$, we must eliminate from the set of admissible initial data those data generating waves with $\mathrm{Im}(k_j) > 0$. There are n parameters to be varied in the initial data. From these, one unimportant parameter, defining the phase shift of the solution, must be dropped, leaving $n - 1$ variable parameters. Let the dynamical system have the following property: when the initial data are prescribed in the neighbourhood of an equilibrium point, there is a point x_{s2} such that one of the anti-symmetric unknowns $(u_i \to u_i, x \to -x)$ equals zero. In a second-order dynamical system, or a fourth-order dynamical system in which the additional terms have small coefficients, this will be the case if there is a second equilibrium point at which there are real values of k, or, what is the same, if the straight line corresponding to this equilibrium point intersects the DC. Then the $n - 1$ variable parameters in the initial data are sufficient to equate the remaining $n - 1$ anti-symmetric unknowns to zero. By virtue of the symmetry of the system in the domain $x > x_{s2}$, we will have symmetry with the solution when $x < x_{s2}$ and the solution will accordingly tend to the first equilibrium points as $x \to +\infty$, hence there will be a solution of the solitary-wave type.

In this analysis the DC has been examined for only one side of the jump. On the boundary of the WZ there is a solitary wave, that is, an essentially non-linear formation; formally, therefore, it is not possible to use the WZ obtained by linearization of the initial equations for the other side. We can agree to associate the WZ for the other side of the jump with the second equilibrium point (shown in

Fig. 1 by the dashed curve). Described in this crude manner, without allowing for the presence of a WZ, the jump turns out to be evolutional. As is obvious from Fig. 1, version 1, the straight line corresponding to the phase velocity for different sides of the jump will lie on different sides of the dispersion curve. This means that the corresponding characteristic on different sides of the jump will be an incoming one, and for the case of the simplified equation with one unknown it will suffice to have one boundary condition, such as $U = c + U(\Delta u)$, where c is the characteristic velocity for state 1 and Δu is the amplitude of the jump.

For a more detailed analysis, generalizing the results for Schrödinger's equation [5], the evolutionality of these jumps may be interpreted as follows. We have p characteristics on one side of the jump, say side 1, and $p + 2$ on the other, say side 2. For the first p characteristics on side 2, the characteristic velocities are the same as on side 1. The two last characteristics on side 2 are parallel to the line of discontinuity. One of them must be considered as incoming from the wave zone; the other, forming the fan of characteristics of a centred simple wave for the averaged system, must be considered as outgoing. We have p obvious conditions, expressing the fact that the parameters are the same on both sides of the solitary wave (in [5] these conditions are expressed as conservation laws). Instead of these conditions, one can use p conservation conditions for the Riemann invariants, if such exist. Since these p conditions were already taken into account in the single integration of the system of equations for steady-state solutions, we may call them principal conditions, by analogy with the theory of dissipative jumps. In addition, two further conditions, determined by examining the discontinuity structure, are needed for evolutionality. They are the amplitude (the amplitude of the solitary wave) and the wavelength (for a sequence of solitary waves the wavelength tends to infinity) in domain 2.

Remark. The criterion indicated above for the existence of a solitary wave also holds for the so-called 1:1 soliton (an envelope soliton whose velocity is identical with the phase velocity of the waves). It may exist in fourth-order systems which have an equilibrium point of the generalized focus type (four complex values $k_{1,2,3,4} = \pm k \pm i\lambda$), and it consists of two waves with the same space period, interacting in resonance. The existence of a second equilibrium point is not required here. One wave increases as $x \to +\infty$, and the other decreases. Owing to energy exchange between the waves, one also obtains solution of the envelope solitary-wave type, of order $\exp(\lambda x)\sin(kx)$ as $x \to -\infty$ and $\exp(-\lambda x)\sin(kx)$ as $x \to +\infty$. In higher-order systems, multiwave analogues are possible. Besides the 1:1 soliton for generalized Korteweg–de Vries (KdV) equations (see below, (3.5)) and generalized Schrödinger equations, there are denumerably many series of solitary waves of a more complicated form.

2. *A jump without radiated waves (pure jump) but with an additional condition (kink).* Suppose there are two equilibrium points C_1 and C_2 for which the corresponding straight lines do not intersect the DC. There are no real roots. For both equilibrium points, there are n values of k with $\mathrm{Im}(k) > 0$ and n values of k with $\mathrm{Im}(k) < 0$. Consider the subspaces $S_1 = S(C_1, \varepsilon_1; \mathbf{u} \to C_1, x \to -\infty)$ and $S_2 = S(C_2, \varepsilon_2; \mathbf{u} \to C_2, x \to +\infty)$. Both subspaces are of dimension n. However, they are situated in a special way relative to each other. If they have a point of intersection, this means—since these are spaces of phase curves of the same dynamical system—that there is a common tangent at that point (phase subspaces), and therefore also a curve of intersection, to which there corresponds a kink-type solution. In that case, if the dimensions of the subspaces add up to that of the whole space, the standard position is that in which they do not intersect. But there may be an intersection in a special case, that is, when a further condition is imposed on the coefficients of the dynamical system.

An algorithmic approach. For both equilibrium points, there are n increasing waves. Suppose the phase curves leaving the neighbourhood of the initial equilibrium point pass near the final equilibrium point. For a *KdV* equation with cubic non-linearity, and its analogue with a fifth-order derivative, this is the case if there is a third stable equilibrium point between the previous two, and the phase portrait is nearly symmetric. Then $n - 1$ variable parameters in the neighbourhood of the initial equilibrium point are not enough to make the amplitude of the n increasing waves vanish in the neighbourhood of the second equilibrium point. One more parameter is needed, possibly the variation of the coefficients of the dynamical system, leading to the appearance of an additional condition at the jump.

Since the straight line corresponding to the phase velocity lies on the same side of the dispersion curve on both sides of the jump, it follows that the corresponding characteristic on one side of the jump is incoming, while that on the other side is outgoing; hence a further boundary condition is necessary.

Remarks. 1. This type of jump may also exist in dissipative systems, since weak dissipation does not affect the sign of the imaginary part of k at unstable equilibrium points, and it also slightly influences the nature of the phase portrait.

2. Solutions of this type also exist when the simplified system is not hyperbolic [6] (a system with complex characteristic values), in which case "non-intersection" should imply "no real roots at $\omega = 0$". Numerical experiments have demonstrated the existence of jumps of this type for a generalized Schrödinger equation, when there is an unstable uniform state on one side of the jump (non-hyperbolicity) and a stable one on the other.

The minimum order of a dynamical system that may have solutions of the type described above is two. In what follows we will consider solutions for which the minimum order is four.

3. *A jump with radiation*. Suppose there are two equilibrium points. For one of them (C_1) the straight line $\omega = kU$ does not intersect the dispersion curve, but for the other (C_2) it intersects it once. At C_1 there are n increasing waves and at C_2 there are $n - 1$ increasing waves. Consider the subspaces $S_1 = S(C_1, \varepsilon_1; \mathbf{u} \to C_1, x \to -\infty)$ and $S_2 = S(C_2, \varepsilon_2; \mathbf{u}(x + T) \to \mathbf{u}(x), x \to +\infty)$. These are phase spaces of dimensions n and $n + 1$, respectively, adding up to $2n + 1$. In this case the standard position is the existence of one, several or denumerably many curves of intersection. Corresponding to the intersection curve is a jump with radiation.

The algorithmic approach. Suppose that there is a set of initial data near the first equilibrium point for which the phase curve passes near the second equilibrium point. Then $n - 1$ variable parameters near the initial equilibrium point (the phase shift of the solution is an unimportant parameter) are sufficient to make the amplitude of $n - 1$ increasing waves vanish in the neighbourhood of the final point and to obtain the required solution. During the solution of the problem, one determines two additional conditions for the jump: the amplitude and wavelength of the radiated wave.

Since the dispersion curve in the neighbourhood of the origin lies below the straight line $\omega = Uk$ on one side of the jump and above it on the other, it follows that, as in the case of jumps of the ordinary type in dissipative systems, the number of outgoing principal characteristics is one less than the number of incoming principal characteristics. The jump is evolutional, since the two additional boundary conditions define two additional outgoing characteristics associated with the radiated wave.

Remark. Numerical solutions of the problem of the attenuation of an initial discontinuity for the generalized *KdV* and Schrödinger equations show that such a jump will appear in fourth-order dynamical systems if the point C_1 is such that all corresponding k values are complex. This may be explained by comparing the phase portraits for the modified *KdV* equation with the fifth derivative in the section of the phase space (a, a_x), $a_{xx} = 0$, $a_{xxx} = 0$ at $x = 0$ in this case and in the case that all the values of k are imaginary. In the case of complex k values, the phase curves leaving from C_1 are spiral-shaped and enclose the whole space around that point. In the case of imaginary k values, generalized separatrices are formed, and the point C_2 is left, as it were, in a "pocket": the phase curves cannot reach it, and therefore there cannot possibly be a solution that is a jump with radiation.

We have considered solutions 1, 2, 3 which, in the context of this analysis, may be called "primary" solutions. In what follows we will consider solutions that may be classed as secondary, that is, obtained by superimposing an additional wave.

1a. *A solitary wave with a superimposed periodic wave (a generalized solitary wave)*. Suppose the straight line corresponding to the equilibrium point C_1 intersects the dispersion curve once. There are $n - 1$ values of k with $\mathrm{Im}(k) < 0$ and $n - 1$ values with $\mathrm{Im}(k) > 0$. Consider the subspaces $S_1 = S(C_1, \varepsilon; \mathbf{u}(x + T) \to \mathbf{u}(x), x \to -\infty)$ and $S_2 = \{\mathbf{u}, u_{2i} = 0\}$. They are of dimensions $n + 1$ and n, respectively, adding up to $2n + 1$, that is, one more than that of the space. The general position is the existence of finitely or denumerably many single-parameter families of intersections between the subspaces. To each intersection point there corresponds a curve of the phase space describing a solitary wave with a superimposed periodic wave.

The algorithmic approach. As before, the number of variable parameters for the initial data is $n - 1$. Here there are $n - 1$ values of k with positive imaginary part, but, unlike the previous case, the phase shift of the increasing waves relative to the periodic component is an essential parameter. The solutions contain one free (undefined) parameter: the amplitude of the superimposed wave. Nevertheless, the problem of finding a solution in which the superimposed wave is of minimum amplitude may be formulated in a well-posed manner.

Numerical experiments for the modified Schrödinger and *KdV* equations demonstrate that a quasi-steady-state solution which can be treated as a solitary wave with superimposed periodic wave appears when there is an initial jump of small amplitude. It is impossible to analyse for evolutionality within the framework of simple hyperbolic systems, since there is a complicated non-local process of energy exchange between long and short waves. By induction, for systems of order greater than four there may be solutions which can be treated as solitary waves with m superimposed waves, if the straight line intersects the dispersion curve m times. These solutions contain m free parameters. The problem of finding a pure solitary wave under these conditions is not well-posed, since the condition that there must be no periodic component reduces the number of variable parameters. Nonetheless, in special cases, when the system divides into blocks, a solitary wave is possible.

2a and 3a. *A kink with radiation and a jump with two radiated waves*. We will now superimpose waves on non-symmetric solutions of types 2 and 3. Suppose that there is one intersection on either side of the jump (the order of the system is greater than 2), or one intersection on one side and two on the other (the order of the system is greater than four). Then we may have solutions which are jumps with an additional condition and a superimposed wave or jumps with radiation and a superimposed wave. The solutions contain two undefined parameters: the amplitude and phase shift of the superimposed wave relative to the jump (obviously, in the first case the additional condition is retained, for otherwise there would be only one undefined parameter). We have phase subspaces $S_1 = S(C_1, \varepsilon_1; \mathbf{u}(x+T) \to \mathbf{u}(x), x \to -\infty)$ and $S_2 = S(C_2, \varepsilon_2; \mathbf{u}(x+T) \to \mathbf{u}(x), x \to +\infty)$ (2a) or $S_2 = S(C_2, \varepsilon_2; |\mathbf{u}(x)| < M, x \to +\infty)$ (3a). They are of dimensions $n+1, n+1$ or $n+1, n+2$, respectively, for a total dimension of $2n+2$ or $2n+3$.

The presence of the two new free parameters implies that the problem of finding a solution in which the superimposed wave is only on one side is well-posed. Its phase, amplitude and therefore also wavelength are well-defined. In the first case we have phase spaces $S_1 = S(C_1, \varepsilon_1; \mathbf{u} \to C_1, x \to -\infty)$ and $S_2 = S(C_2, \varepsilon_2; \mathbf{u}(x+T) \to \mathbf{u}(x+T) \to \mathbf{u}(x), x \to +\infty)$. The dimensions of the subspaces are $n-1$ and $n+1$, adding up to $2n$. In the second case we have subspaces $S_1 = S(C_1, \varepsilon_1; \mathbf{u} \to C_1, x \to -\infty)$ and $S_2 = S(C_2, \varepsilon_2; |\mathbf{u}(x)| < M, x \to +\infty)$ of dimensions $n+1$ and n, respectively, adding up to $2n+1$. This gives structures of evolutional jumps with three and four additional conditions, respectively.

Proceeding in an analogous way by induction for higher-order systems, one can similarly superimpose m waves. If the straight line intersects the dispersion curve m times on either side of the jump, structures of evolutional jumps with m radiated waves (each only on one side) and $2m+1$ additional conditions may appear. If there are m intersections on one side and $m+1$ on the other, an evolutional-jump structure with $2m+2$ additional conditions and m radiated waves may appear. By radiated waves on the right of the jump (in the domain $x > x_0$) we mean waves for which $c_g = d\omega/dk > U$, while those on the left have $c_g < U$. In the opposite case, a wave on either side of the jump is said to be absorbed. It is assumed here that the characteristic velocities generated by a radiated wave may be expressed as $c = c_g + f(a)$, as in the case, e.g. for weakly non-linear waves described by the non-linear Schrödinger equation. If the wave amplitude is not high, then $\operatorname{sign}(c_g - U) = \operatorname{sign}(c - U)$. For this reason, cases of tangency to the dispersion curve, when $U = c_g$, are excluded from consideration in connection with evolutionality. The case in which a radiated wave generates one outgoing and one incoming characteristic does not arise in numerical computations and will not be considered here. This apparently produces a situation in which the jump is not evolutional and is replaced with a transient solution or a jump of the soliton type. Thus, the waves are assumed to be weakly non-linear, although numerical experiments show that these conclusions are valid for finite-amplitude waves.

The most general formulation of the condition for a jump structure to exist. Suppose that there are n_{1il} and n_{2ab}, respectively. The total number of intersections with the dispersion curves on both sides of the jump is $n_{ab} + n_{il}$. Then a necessary condition for an evolutional steady-state structure with only radiated waves to exist is

$$(-n_{1ab} + n_{1il}) + (-n_{2ab} + n_{2il}) + n_h/2 = 0 \tag{2.1}$$

where n_h is the difference between the number of outgoing and incoming characteristics, not counting the characteristics of radiated waves (there are two outgoing characteristics associated with each radiated wave). The total number of additional boundary conditions at the jump is $n_h/2 + 2n_{1il} + 2n_{2il} + 1$. The simplest non-standard case of intersection with the DC is that in which there is a sector of the DC that is intersected twice on one side of the jump and never on the other side. In that case, one of the waves corresponding to these intersections is radiated, and the other, absorbed. By (2.1), an evolutional structure of analogues of the evolutional jumps considered above—either of kinks ($n_h = 0$), or of ordinary jumps ($n_h = -2$), may exist. Generalizations to the case of several such sectors are obvious. In fact, the case of jumps whose velocity is close to the characteristic velocity $c = d\omega/dk(0)$ (low-amplitude jumps) has been thoroughly investigated. By adding a radiated wave on one side of the jump and an absorbed wave on the other to the solutions of 2 and 3, or a radiated wave and an absorbed wave on one side of the jump, one can investigate the evolutionality of all possible types of jump with a non-soliton structure.

3. NUMERICAL VERIFICATION OF THE THEORY

Using the symmetry properties of the equations, an efficient numerical method has been evolved to verify the existence of jumps as conjectured above by numerical solution of the problem of an attenuating discontinuity. The main idea of the method is as follows. Using numerical schemes with central differences in the case when the time step Δt is much less than the spatial step Δx, a numerical solution may be interpreted as the solution of a certain new system, including terms with derivatives of higher order with coefficients proportional to Δx^2. It is important that this new system belong to the class of conservative symmetric systems considered above. To that end, conservative numerical schemes must be used. This makes it possible to predict possible scheme effects, and one obtains numerical solutions with highly reliable qualitative properties [6].

As illustrations pertaining to KdV-like models, consider the following

$$a_t - (a^3)_x + b_3 a_{xxx} = 0, \{q_i\} = (a, a_x) \tag{3.1}$$

$$a_t + a a_x + b_3 a_{xxx} + b_5 a_{xxxxx} = 0, \{q_i\} = (a, a_x, a_{xx}, a_{xxx}) \tag{3.2}$$

$$a_t - (a^3)_x + b_3 a_{xxx} + b_5 a_{xxxxx} = 0, \{q_i\} = (a, a_x, a_{xx}, a_{xxx}) \tag{3.3}$$

Equation (3.3) was specially constructed to verify the theory; Eqs (3.1) and (3.2) are used to describe phenomena in plasma and to describe waves on water with an ice or some other elastic cover.

Figure 3 shows a graph of a solution of Eq. (3.1) with a jump having a structure of the solitary-wave type and with a jump without radiation (kink); $b_3 = 1$. Figure 4 shows graphs of solutions with a jump with a radiated wave for Eq. (3.2); $b_3 = -5$ (curve 1), $b_3 = 1$, $b_5 = 4$ (curve 2). Figure 5 shows a graph of a solution of Eq. (3.3) of the kink type with radiation (see 3a in the previous section); $b_3 = 1$, $b_5 = 1$.

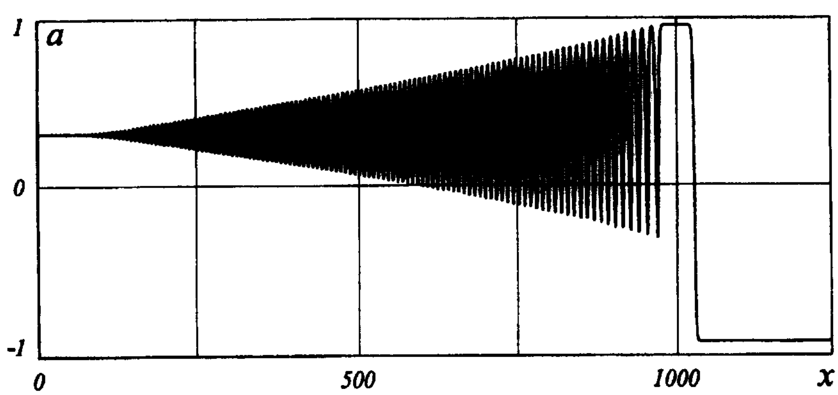

Fig. 3.

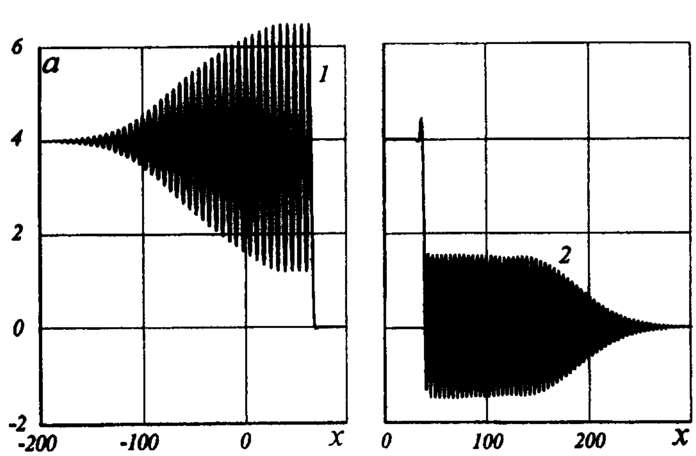

Fig. 4.

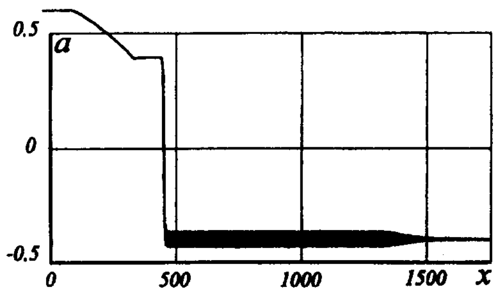

Fig. 5.

Computations have shown that the theory also holds when the DC has several branches. Investigations were conducted for waves whose envelope was described by a generalized Schrödinger equation with a third-order derivative [4, 6] (jumps with solitary-wave type structure, a jump with a radiated wave, and pure jumps without radiation were found), and for a cold plasma with high-order dispersion.† In all these models there is one additional conservation law.

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†BAKHOLDIN, I. B., Modelling of the transient evolution of solitary waves. Preprint No. 61. Moscow, Inst. Prikl. Mat. im. M. V. Keldysh, Ross. Akad. Nauk, 1997. This paper investigated the problem of the evolution of initial data of the solitary-wave type, but jumps with a structure of the solitary-wave type and jumps with radiation have now been obtained.